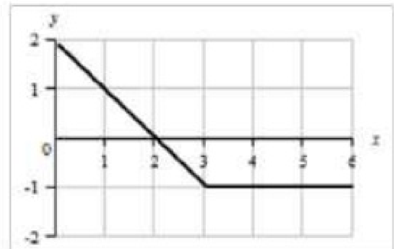
Chapter 6: Fundamental Theorem of Calculus

I. Fundamental Theorem of Calculus

1) $\boxed{H(x) = \int_0^x g(t)\,dt}$. The graph of g(x) is given, use it to answer the following questions:



graph of g

$H' = g$

$H'' = g'$

Justify all answers

a) On what interval is H increasing?

$V(0) = 2$
Since $V(0) > 0$
H is increasing at $t = 0$

$[0, 2)$ b/c $H' > 0$

b) On what interval is H decreasing?

$V(6) = -1$
Since $V(6) < 0$
H is decreasing at $t = 6$

$(2, 6]$ b/c $H' < 0$

c) Where is H concave down?

$(0, 3)$ b/c $H'' < 0$

d) Is H(6) positive or negative?

$H(6) = \int_0^6 g(t)\,dt = \frac{1}{2}(2)(2) - \frac{1}{2}(1)(4+3)$
$= -1\frac{1}{2}$

e) Find the equation of the tangent to H at $x = 4$

point: $(4, H(4))$

slope: $H'(4) = g(4) = -1$

$$H(4) = \int_0^4 g(t)\,dt$$
$$= \frac{1}{2}(2)(2) - \frac{1}{2}(1)(1+2)$$
$$= 2 - \frac{3}{2}$$
$$= \frac{1}{2}$$

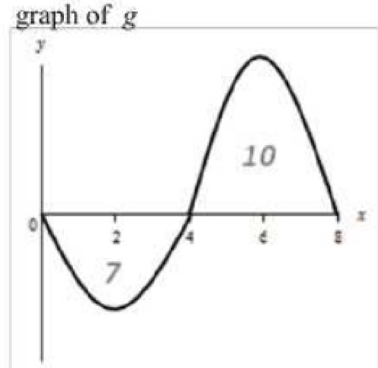
$(4, \frac{1}{2})$

$$\boxed{y - \frac{1}{2} = -1(x - 4)}$$

2) $H(x) = \int_0^x g(t)\,dt$. The graph of g(x) is given, use it to answer the following questions:

Justify all answers

graph of g



a) On what interval is H increasing?

$(4,8]$ b/c $H' > 0$

$(4,8)$

b) On what interval is H decreasing?

$[0,4)$ b/c $H' < 0$

c) Where is H concave down?

$(0,2)$ $(6,8)$ b/c $H'' < 0$

d) Is H(8) positive or negative?

H(8) is positive b/c $H(8) = \int_0^8 g(t)\,dt = -7 + 10 = 3$

$H' = g$

$H'' = g'$

e) Find the equation of the tangent to H at $x = 8$

point

$(8, H(8))$

$(8, 3)$

slope

$H'(8) = g(8) = 0$

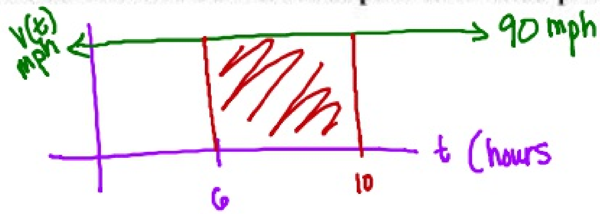
$y - 3 = 0(x - 8)$

$y = 3$

Last night I was really, really bored. I went down to Southtowne Mall and I noticed the most amazing thing. It was 6:00 p.m. and on I-15 there was no traffic – only one lone car. With my miraculous skills at estimating rates I determined the car was traveling at 90 mph.
(It was an amazing car that drove all that day and all through the night with the cruise control on.)

Determine how far the car traveled between 6:00 p.m. and 10:00 p.m.

Draw a picture:



Write a function to determine the distance traveled for ***any length of time*** relative to when I began observation.

$$d(t) = \int_6^t 90\,dx$$

Using the function from above, determine the distance the car traveled between 6:00 and 10:00 p.m.

$$d(10) = \int_6^{10} 90\,dx$$

Evaluate d(10) both geometrically and using an antiderivative.

<u>Distance traveled using geometry</u>

$$d(10) = \int_6^{10} 90\,dx$$

$$= (4\,\cancel{\text{hours}})\left(90\,\frac{\text{miles}}{\cancel{\text{hours}}}\right)$$

$$= 360 \text{ miles}$$

<u>distance traveled using an antiderivative</u>

$$d(10) = \int_6^{10} 90\,dx$$

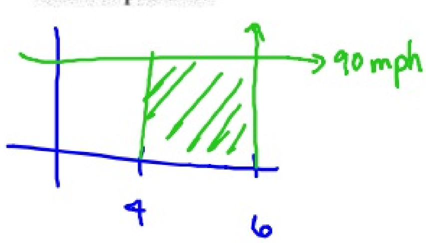
$$= 90x \Big|_6^{10}$$

$$= 90(10) - 90(6)$$

$$= 360 \text{ miles}$$

Using d(t), determine how far the car traveled between 4:00 p.m. and 6:00 p.m.

Draw a picture:



Determine the distance: (use integral notation)

$$d(4) = \int_6^4 90\,dx$$

$$= -\int_4^6 90\,dx$$

$$= -90x \Big|_4^6$$

$$= -90(6) - {-90(4)}$$

$$= -180 \text{ miles}$$

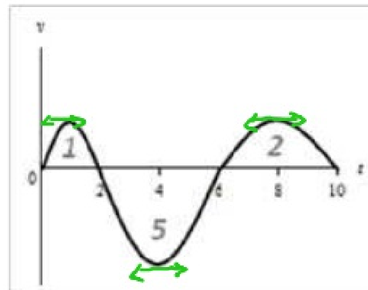
What does the negative sign represent in the answer?

distance before observation

3) The position of an object moving forward and backward at any time t is given by the function $s(t) = \int_2^t v(x)dx$, the velocity of the object is shown in the graph. *Justify all answers.*

$s' = v$
$s'' = v' = a$

graph of v



a) When is the object moving forward?

$(0,2)\ (6,10)$ b/c $v > 0$

b) When is $a(t) = 0$?

$t = 1, 4, 8$ b/c $v' = 0$

c) When is the object slowing down?

$(1,2)\ (4,6)\ (8,10)$ b/c v + a are opposite signs

d) What is the position of the object at 2 seconds, 0 second, 6 seconds, and 10 seconds?

$s(2) = \int_2^2 v(x)dx = 0$

$s(0) = \int_2^0 v(x)dx = -\int_0^2 v(x)dx = -1$

$s(6) = \int_2^6 v(x)dx = -5$

$s(10) = \int_2^{10} v(x)dx = -5 + 2 = -3$

e) Find when the object is farthest to the right.

v: $+$ on $(0,2)$, $-$ on $(2,6)$, $+$ on $(6,10)$

0 min, 2 max, 6 min, 10 max

$s(2) = 0$
$s(10) = -3$

The object is farthest right at $t = 2$.

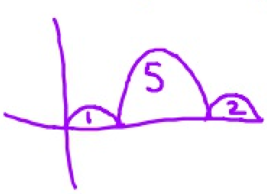
h) Find the displacement of the object on [2, 10]

$$\int_2^{10} v(x)dx = s(x)\Big|_2^{10} = s(10) - s(2) = (-3) - 0 = -3$$

i) Find the total distance traveled by the object on [2, 10] *(Draw the picture of what you are integrating)*

$|v(x)|$

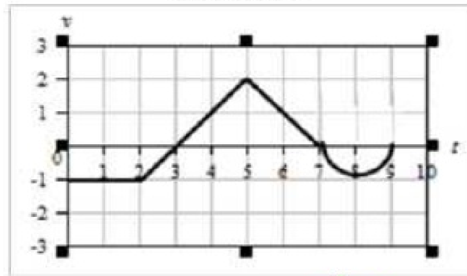
$$\int_2^{10} |v(x)|dx = 5 + 2 = 7$$



4) The position of an object moving left and right at any time t is given by the function

$s(t) = \int_3^t v(x)dx$, the velocity of the object is shown in the graph.

graph of v



Justify all answers.

a) When is the object moving left?

$[0,3)\ (7,9)$ b/c $v<0$

b) When does the object change directions?

$t=3,7$ b/c v changes signs

$s'=v$
$s''=v'=a$

c) When is the object slowing down?

$(2,3)\ (5,7)\ (8,9)$ b/c v & a are opposites

d) What is the position of the object at 3 seconds, 0 second, 5 seconds, 7 seconds and 9 seconds?

$s(3) = \int_3^3 v(x)dx = 0$

$s(0) = \int_3^0 v(x)dx = -\int_0^3 v(x)dx = -\left[-\left(\frac{1}{2}\right)(1)(3+2)\right] = \frac{5}{2} = 2.5$

$s(5) = \int_3^5 v(x)dx = \frac{1}{2}(2)(2) = 2$

$s(7) = \int_3^7 v(x)dx = \frac{1}{2}(4)(2) = 4$

$s(9) = \int_3^9 v(x)dx = 4 - \frac{1}{2}(\pi)(1)^2 = 4 - \frac{\pi}{2}$

e) Find when the object is farthest to the right *(hint: max. value)*.

v [− | + | −]
0 3 7 9
max min max min

$S(0) = 2.5$
$S(7) = 4$

The object is farthest right at $t=7$

f) Find when the object is farthest to the left *(hint: min. value)*.

$S(3) = 0$
$S(9) > 0$

The object is farthest left @ $t = 3$.

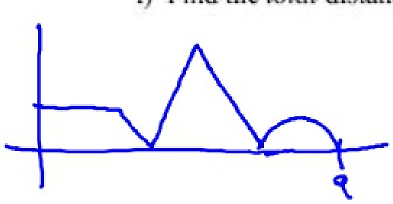
g) Is the acceleration of the object at 6 sec. positive, negative or zero?

$a(6) = v'(6) = -1$ b/c $v'(6) = -1$

h) Find the *displacement* of the object on [3, 9]

$\int_3^9 v(x)\,dx = s(t)\Big|_3^9 = s(9)-s(3) = \left(4-\frac{\pi}{2}\right) - 0 = 4 - \frac{\pi}{2}$

i) Find the *total* distance traveled by the object on [3, 9]. *(Draw the picture of what you are integrating)*



$\int_3^9 |v(x)|\,dx = 4 + \frac{\pi}{2}$

II. Tables

5) The tables below shows the velocity of dollars earned per day at a lemonade stand. Use the table to answer the following:

L(t) =

days	$/day
0	100
1	120
2	150
3	200
4	160
5	165
6	150
7	130
8	90

a) Find the average acceleration over $0 \le t \le 8$.

$$\frac{L(8)-L(0)}{8-0} = \frac{90-100}{8} = \frac{-10}{8} = -\frac{5}{4}\ \$/day^2$$

b) Estimate the acceleration at day 5.

$$\frac{L(6)-L(4)}{6-4} = \frac{150-160}{2} = -5\ \$/day^2$$

c) When is $a(t) < 0$

$(3,4)$ $(5,8)$ b/c $L(t)$ is decreasing

$L'(t) < 0$

d) Estimate the total dollars earned in $0 \le t \le 8$ using a left rectangular approximation. (8 subdivisions)

$$\text{total dollars earned} = \int_0^8 L(t)\,dt$$

$$\approx (1)\left[L(0)+L(1)+L(2)+L(3)+\cdots+L(7)\right]$$

$$\approx (1)\left[100+120+150+200+160+165+150+130\right]$$

$\approx$ 1,175 dollars earned in the first 8 days

e) Find the average dollars earned *(velocity)* for the $0 \le t \le 8$.

$$\frac{1}{b-a}\int_a^b f(t)\,dt$$

$$\text{Ave \$ per day} = \frac{1}{8-0}\int_0^8 L(t)\,dt \approx \frac{1}{8}(1{,}175) \approx 146.875$$

$\approx \$146.88$

f) Use the trapezoid rule to estimate the dollars earned during $0 \le t \le 4$. (4 subdivisions)

$$\text{total dollars earned} = \int_0^4 L(t)\,dt$$

$$\approx \frac{1}{2}(1)\left[L(0)+2\cdot L(1)+2\cdot L(2)+2\cdot L(3)+L(4)\right]$$

$$\approx \frac{1}{2}\left[100+2(120)+2(150)+2(200)+160\right]$$

$$\approx \frac{1}{2}\left[1200\right]$$

$\approx \$600$ earned during days $0 \le t \le 4$

6) Chocolate was flowing out of a large tank at varying rates. The rate at which the chocolate was flowing out was measured every 3 minutes and the rates are given in the chart below. Use the chart to answer the following questions.

TABLE

	C(m)
minutes	**gal./min.**
0	0
3	2
6	5
9	4
12	3
15	1
18	3
21	6
24	9

a) When was the chocolate flow accelerating?

$(0,6)$ $(15,24)$ b/c $C(m)$ is incr.

$C'(m) > 0$

b) What was the average rate of acceleration during $0 \le t \le 24$ mins.?

$$\frac{C(24)-C(0)}{24-0} = \frac{9-0}{24-0} = \frac{3}{8}\ \text{gal}/\text{min}^2$$

c) Use a MRAM to estimate the amount of chocolate drained during $0 \le t \le 24$ minutes? *(4 subdivisions)*

$$\text{Amount of Chocolate drained} = \int_0^{24} C(m)\,dm$$

$$\approx 6\left[C(3) + C(9) + C(15) + C(21)\right]$$

≈ 78 gallons of chocolate

d) What is the average rate of flow over $0 \le t \le 24$ minutes?

$$\text{Ave. flow rate} = \frac{1}{24-0}\int_0^{24} C(m)\,dm$$

$$\approx \frac{1}{24}(78)$$

$$\approx 3.25\ \text{gal}/\text{min.}$$

e) Use the trapezoid rule to estimate the amount of chocolate drained during $0 \le t \le 15$ mins. *(5 subdivisions)*

$$\text{Amount of chocolate drained} = \int_0^{15} C(m)\,dm$$

$$\approx \frac{1}{2}(3)\left[C(0) + 2\cdot C(3) + \cdots + 2\cdot C(12) + C(15)\right]$$

$$\approx \frac{3}{2}\left[0 + 2\cdot 2 + 2(5) + 2(4) + 2(3) + 1\right]$$

$\approx \frac{87}{2}$ gallons

≈ 43.5 gallons during 0 to 15 mins.