Day 1: Disc/Washer

Section 6.2 (page 420)

1. $\frac{\pi}{3}$ 3. $\frac{16\pi}{3}$ 5. $\frac{15\pi}{2}$ 7. $\frac{2\pi}{35}$ 9. 8π

11. $\frac{\pi}{4}$ 13. a. 8π b. $\frac{128\pi}{5}$ c. $\frac{256\pi}{15}$ d. $\frac{192\pi}{5}$

15. a. $\frac{32\pi}{3}$ b. $\frac{64\pi}{3}$ 17. 18π

19. $\pi(8\ln 4 - \frac{3}{4}) \approx 32.49$

21. $\frac{208\pi}{3}$ 23. $\frac{384\pi}{5}$ 25. $\pi \ln 4$ 27. $\frac{3\pi}{4}$

29. $\frac{\pi}{2}\left(1 - \frac{1}{e^2}\right) \approx 1.358$ 31. π 33. a

35. a. $\frac{512\pi}{15}$
b. No, the solid has only been translated horizontally.

37. 18π 41. $\pi r^2 h\left(1 - \frac{h}{H} + \frac{h^2}{3H^2}\right)$ 43. $\frac{\pi}{30}$

45. One-fourth: 32.64 ft
Three-fourths: 82.64 ft

47. 1.969 49. 49.023

51. a. $\frac{128}{3}$ b. $\frac{32\sqrt{3}}{3}$ c. $\frac{16\pi}{3}$ d. $\frac{32}{3}$

53. a. $\frac{1}{10}$ b. $\frac{\pi}{80}$ c. $\frac{\sqrt{3}}{40}$ d. $\frac{3}{80}$ e. $\frac{\pi}{20}$

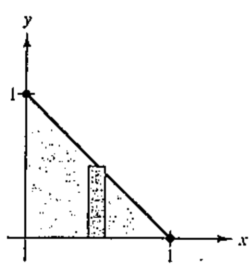
55. a. $\frac{2r^3}{3}$ b. $\frac{2r^3 \tan\theta_0}{3}$ 59. $5\sqrt{1 - 2^{-2/3}} \approx 3.0415$

EXERCISES for Section 6.2

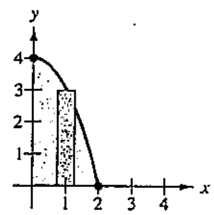
TECHNOLOGY Laboratory Guide Labs 6.2–6.3

In Exercises 1–8, find the volume of the solid formed by revolving the region about the *x*-axis.

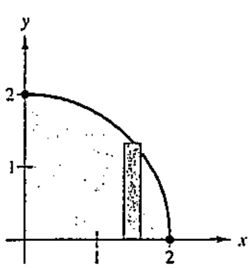
1. $y = -x + 1$



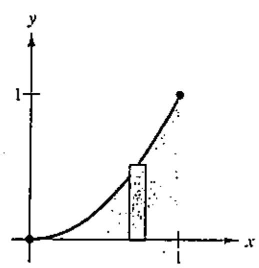
2. $y = 4 - x^2$



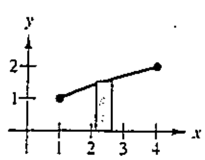
3. $y = \sqrt{4 - x^2}$



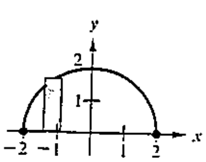
4. $y = x^2$



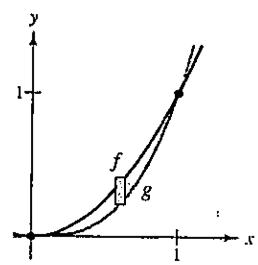
5. $y = \sqrt{x}$



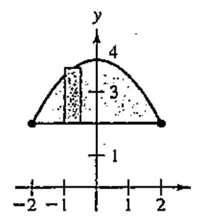
6. $y = \sqrt{4 - x^2}$



7. $y = x^2$, $y = x^3$

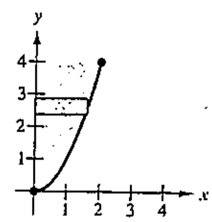


8. $y = 2$, $y = 4 - \frac{x^2}{2}$

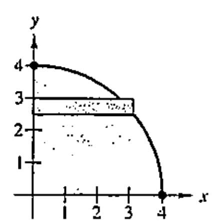


In Exercises 9–12, find the volume of the solid formed by revolving the region about the *y*-axis.

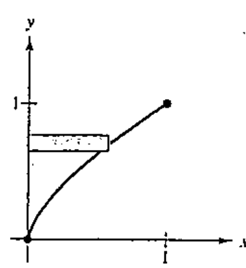
9. $y = x^2$



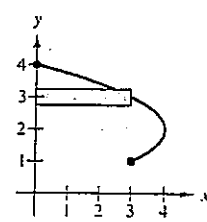
10. $y = \sqrt{16 - x^2}$



11. $y = x^{2/3}$



12. $x = -y^2 + 4y$



Representative rectangle must be perpendicular to the axis of rotation

In Exercises 13–16, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the indicated lines.

13. $y = \sqrt{x}$, $y = 0$, $x = 4$
  a. the x-axis b. the y-axis
  c. the line $x = 4$ d. the line $x = 6$

14. $y = 2x^2$, $y = 0$, $x = 2$
  a. the y-axis b. the x-axis
  c. the line $y = 8$ d. the line $x = 2$

15. $y = x^2$, $y = 4x - x^2$
  a. the x-axis b. the line $y = 6$

16. $y = 6 - 2x - x^2$, $y = x + 6$
  a. the x-axis b. the line $y = 3$

In Exercises 17–20, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the line $y = 4$.

17. $y = x$, $y = 3$, $x = 0$ 18. $y = x^2$, $y = 4$

19. $y = \dfrac{1}{x}$, $y = 0$, $x = 1$, $x = 4$

20. $y = \sec x$, $y = 0$, $0 \leq x \leq \dfrac{\pi}{3}$

In Exercises 21–24, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the line $x = 6$.

21. $y = x$, $y = 0$, $y = 4$, $x = 6$

22. $y = 6 - x$, $y = 0$, $y = 4$, $x = 0$

23. $x = y^2$, $x = 4$ 24. $xy = 6$, $y = 2$, $y = 6$, $x = 6$

In Exercises 25–32, find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the x-axis.

25. $y = \dfrac{1}{\sqrt{x + 1}}$, $y = 0$, $x = 0$, $x = 3$

26. $y = x\sqrt{4 - x^2}$, $y = 0$

27. $y = \dfrac{1}{x}$, $y = 0$, $x = 1$, $x = 4$

28. $y = \dfrac{3}{x + 1}$, $y = 0$, $x = 0$, $x = 8$

29. $y = e^{-x}$, $y = 0$, $x = 0$, $x = 1$

30. $y = e^{x/2}$, $y = 0$, $x = 0$, $x = 4$

31. $y = \sqrt{\sin x}$, $y = 0$, $x = 0$, $x = \dfrac{\pi}{2}$

32. $y = \sqrt{\cos x}$, $y = 0$, $x = 0$, $x = \dfrac{\pi}{2}$

Approximation **In Exercises 33 and 34, determine which value best approximates the volume of the solid generated by revolving the region bounded by the graphs of the equations about the x-axis. (Make your selection on the basis of a sketch of the solid and *not* by performing any calculations.)**

33. $y = e^{-x^2/2}$, $y = 0$, $x = 0$, $x = 2$
  a. 3 b. -5 c. 10 d. 7 e. 20

34. $y = \arctan x$, $y = 0$, $x = 0$, $x = 1$
  a. 10 b. $\frac{3}{4}$ c. 5 d. -6 e. 15

35. a. The region bounded by the parabola $y = 4x - x^2$ and the x-axis is revolved about the x-axis. Find the volume of the resulting solid.
  b. If the equation of the parabola in part a were changed to $y = 4 - x^2$, would the volume of the solid generated be different? Why or why not?

36. Find the volume of the solid generated if the upper half of the ellipse $9x^2 + 25y^2 = 225$ is revolved about
  a. the x-axis to form a prolate spheroid (shaped like a football).
  b. the y-axis to form an oblate spheroid (shaped like an M&M candy).

37. If the portion of the line $y = \frac{1}{2}x$ lying in the first quadrant is revolved about the x-axis, a cone is generated. Find the volume of the cone extending from $x = 0$ to $x = 6$.

38. Use the disc method to verify that the volume of a right circular cone is $\frac{1}{3}\pi r^2 h$, where r is the radius of the base and h is the height.

39. Use the disc method to verify that the volume of a sphere of radius r is $\frac{4}{3}\pi r^3$.

40. A sphere of radius r is cut by a plane h ($h < r$) units above the equator. Find the volume of the solid (spherical segment) above the plane.

41. A cone with a base of radius r and height H is cut by a plane parallel to and h units above the base. Find the volume of the solid (frustum of a cone) below the plane.

42. The region bounded by $y = \sqrt{x}$, $y = 0$, $x = 0$, and $x = 4$ is revolved about the x-axis.
  a. Find the value of x on the interval $[0, 4]$ that divides the solid into two parts of equal volume.
  b. Find the values of x on the interval $[0, 4]$ that divide the solid into three parts of equal volume.

43. *Volume of a Fuel Tank* A tank on the wing of a jet is formed by revolving the region bounded by the graph of $y = \frac{1}{8}x^2\sqrt{2 - x}$ and the x-axis about the x-axis (see figure), where x and y are measured in meters. Find the volume of the tank.

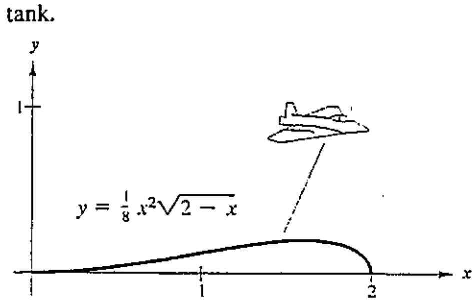


FIGURE FOR 43

Day 2: Shell

Section 6.3 (*page 429*)

1. $\frac{16\pi}{3}$ 3. $\frac{128\pi}{5}$ 5. 8π 7. $\frac{16\pi}{3}$

9. $\frac{8\pi}{3}$ 11. $\sqrt{2\pi}\left(1 - \frac{1}{\sqrt{e}}\right) \approx 0.986$

13. $\frac{8\pi}{3}$ 15. $\frac{\pi}{2}$ 17. 16π 19. 64π

21. a. $\frac{128\pi}{7}$ b. $\frac{64\pi}{5}$ c. $\frac{96\pi}{5}$ d. $\frac{320\pi}{7}$

23. a. $\frac{\pi a^3}{15}$ b. $\frac{\pi a^3}{15}$ c. $\frac{4\pi a^3}{15}$ d. $\frac{4\pi a^3}{15}$ 25. d

27. Diameter $= 2\sqrt{4 - 2\sqrt{3}} \approx 1.464$ 29. $\frac{\pi h^3}{6}$

31. $2\pi^2 r^2 R$

33. Both integrals yield the volume of the solid generated by revolving the region bounded by the graphs of $y = \sqrt{x - 1}$, $y = 0$, and $x = 5$ about the x-axis.

35. 1.5016 37. 186.0552 39. 122,313 ft^3

41. a. $2\pi \int_0^r hx\left(1 - \frac{x}{r}\right) dx$

b. $2\pi \int_{-r}^{r} (R - x)\left(2\sqrt{r^2 - x^2}\right) dx$

c. $2\pi \int_0^r 2x\sqrt{r^2 - x^2}\, dx$ d. $2\pi \int_0^r hx\, dx$

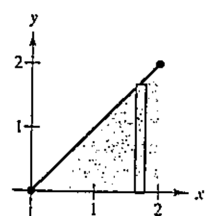
e. $2\pi \int_0^b 2ax\sqrt{1 - \frac{x^2}{b^2}}\, dx$

EXERCISES for Section 6.3

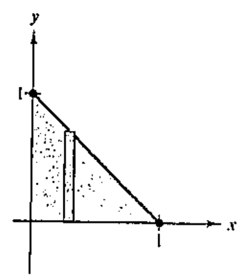
TECHNOLOGY Laboratory Guide Labs 6.2–6.3

In Exercises 1–12, use the shell method to find the volume of the solid generated by revolving the region about the y-axis.

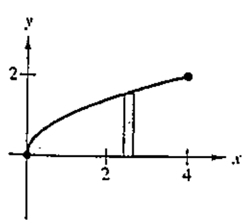
1. $y = x$



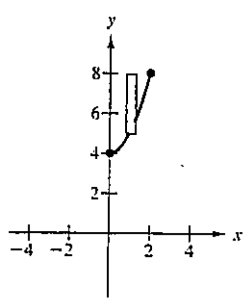
2. $y = 1 - x$



3. $y = \sqrt{x}$



4. $y = x^2 + 4$



5. $y = x^2$, $y = 0$, $x = 2$

6. $y = x^2$, $y = 0$, $x = 4$

7. $y = x^2$, $y = 4x - x^2$

8. $y = 4 - x^2$, $y = 0$

9. $y = 4x - x^2$, $x = 0$, $y = 4$

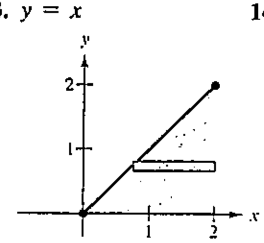
10. $y = 2x$, $y = 4$, $x = 0$

11. $y = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$, $y = 0$, $x = 0$, $x = 1$

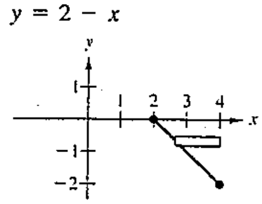
12. $y = \begin{cases} \frac{\sin x}{x}, & x > 0 \\ 1, & x = 0 \end{cases}$, $y = 0$, $x = 0$, $x = \pi$

In Exercises 13–16, use the shell method to find the volume of the solid generated by revolving the region about the x-axis.

13. $y = x$



14. $y = 2 - x$



15. $y = \frac{1}{x}$, $x = 1$, $x = 2$, $y = 0$

16. $x + y^2 = 9$, $x = 0$

Draw the representative rectangle, then use the method appropriate – keep the same direction

In Exercises 17–20, use the shell method to find the volume of the solid generated by revolving the region about the specified line.

17. $y = x^2$, $y = 4x - x^2$, about the line $x = 4$

18. $y = x^2$, $y = 4x - x^2$, about the line $x = 2$

19. $y = 4x - x^2$, $y = 0$, about the line $x = 5$

20. $y = \sqrt{x}$, $y = 0$, $x = 4$, about the line $x = 6$

In Exercises 21–24, use the disc *or* shell method to find the volume of the solid generated by revolving the region bounded by the graphs of the equations about the specified line.

21. $y = x^3$, $y = 0$, $x = 2$
    a. The x-axis b. The y-axis
    c. The line $x = 4$ d. The line $y = 8$

22. $y = \dfrac{1}{x^2}$, $y = 0$, $x = 1$, $x = 4$
    a. The x-axis b. The y-axis
    c. The line $x = 4$ d. The line $y = 1$

23. $x^{1/2} + y^{1/2} = a^{1/2}$, $x = 0$, $y = 0$
    a. The x-axis b. The y-axis
    c. The line $x = a$ d. The line $y = a$

24. $x^{2/3} + y^{2/3} = a^{2/3}$, $a > 0$ (hypocycloid)
    a. The x-axis b. The y-axis

Approximation **In Exercises 25 and 26, determine which value best approximates the volume of the solid generated by revolving the region bounded by the graphs of the equations about the y-axis. (Make your selection on the basis of a sketch of the solid and *not* by performing any calculations.)**

25. $y = 2e^{-x}$, $y = 0$, $x = 0$, $x = 2$
    a. $\frac{3}{2}$ b. -2 c. 4 d. 7.5 e. 15

26. $y = \tan x$, $y = 0$, $x = 0$, $x = \dfrac{\pi}{4}$
    a. 3.5 b. $-\frac{9}{4}$ c. 8 d. 10 e. 1

27. *Machine Part* A solid is generated by revolving the region bounded by $y = \frac{1}{2}x^2$ and $y = 2$ about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one-quarter of the volume is removed. Find the diameter of the hole.

28. *Machine Part* A solid is generated by revolving the region bounded by $y = \sqrt{9 - x^2}$ and $y = 0$ about the y-axis. A hole, centered along the axis of revolution, is drilled through this solid so that one-third of the volume is removed. Find the diameter of the hole.

29. A hole is cut through the center of a sphere of radius r. The height of the remaining spherical ring is h, as shown in the figure. Show that the volume of the ring is $V = \pi h^3/6$. (*Note:* The volume is independent of r.)

30. A **torus** is formed by revolving the region bounded by the circle $x^2 + y^2 = 1$ about the line $x = 2$, as shown in the figure. Find the volume of this "doughnut-shaped" solid. (*Hint:* The integral $\int_{-1}^{1} \sqrt{1 - x^2}\, dx$ represents the area of a semicircle.)

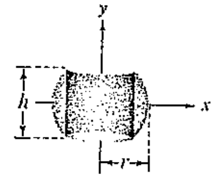


FIGURE FOR 29

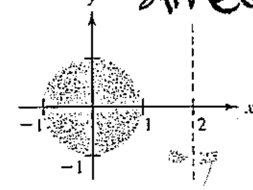


FIGURE FOR 30

31. Repeat Exercise 30 for a torus formed by revolving the region bounded by the circle $x^2 + y^2 = r^2$ about the line $x = R$ where $r < R$.

32. Let a sphere of radius r be cut by a plane, thus forming a segment of height h. Show that the volume of this segment is $\frac{1}{3}\pi h^2(3r - h)$.

In Exercises 33 and 34, write a geometric argument that explains why the integrals have equal values.

33. $\pi \int_1^5 (x - 1)\, dx$, $2\pi \int_0^2 y[5 - (y^2 + 1)]\, dy$

34. $\pi \int_0^2 [16 - (2y)^2]\, dy$, $2\pi \int_0^4 x\left(\dfrac{x}{2}\right) dx$

C **In Exercises 35–38, use Simpson's Rule with $n = 10$ to approximate the volume of the solid. The solid is generated by revolving the region bounded by the graphs of the equations about the y-axis.**

35. $x^{4/3} + y^{4/3} = 1$, $y = 0$, $x = 0$

36. $y = \sqrt{1 - x^3}$, $y = 0$, $x = 0$

37. $y = \sqrt[3]{(x - 2)^2(x - 6)^2}$, $y = 0$, $x = 2$, $x = 6$

38. $y = \dfrac{2}{1 + e^{1/x}}$, $y = 0$, $x = 1$, $x = 3$

39. *Volume of a Storage Shed* A storage shed has a circular base of diameter 80 feet (see figure). Starting at the center, the interior height is measured every 10 feet and recorded in the table. Use Simpson's Rule to approximate the volume of the building.

x	0	10	20	30	40
Height	50	46	40	20	0

40. *Volume of a Pond* A pond is approximately circular with a diameter of 400 feet (see figure). Starting at the center, the depth of the water is measured every 25 feet and recorded in the table. Use Simpson's Rule to approximate the volume of water in the pond.

x	0	25	50	75	100
Depth	20	19	19	17	15

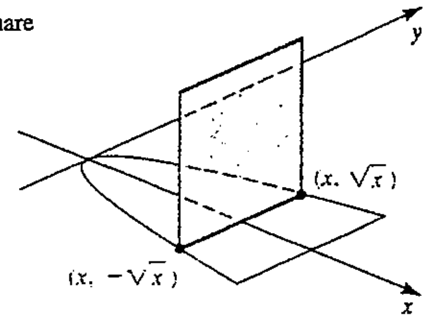
x	125	150	175	200
Depth	14	10	6	0

Known Cross Sections
Day 3

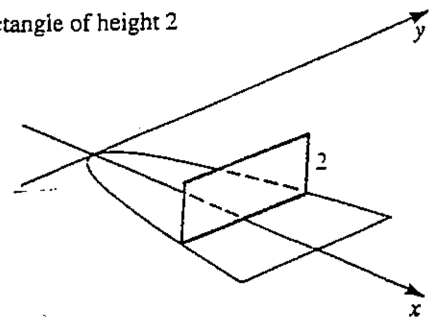
EXERCISES 5.4

Exer. 1–8: Let R be the region bounded by the graphs of $x = y^2$ and $x = 9$. Find the volume of the solid that has R as its base if every cross section by a plane perpendicular to the x-axis has the given shape.

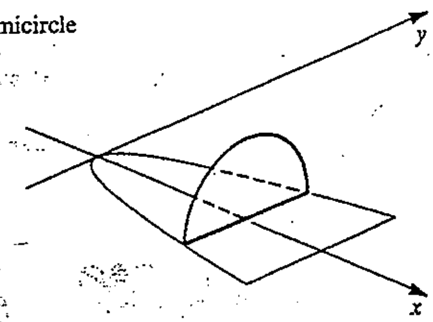
1 A square



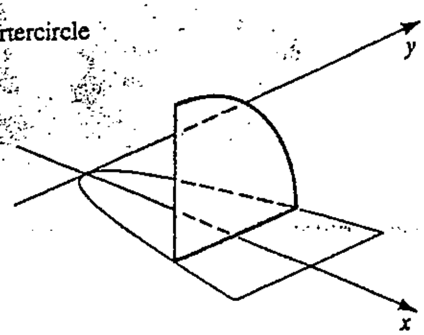
2 A rectangle of height 2



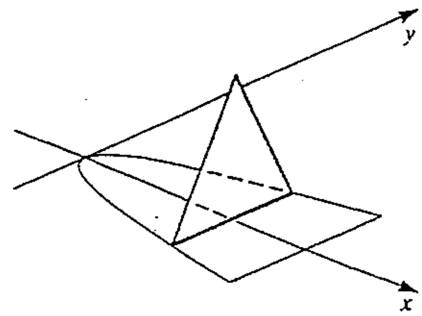
3 A semicircle



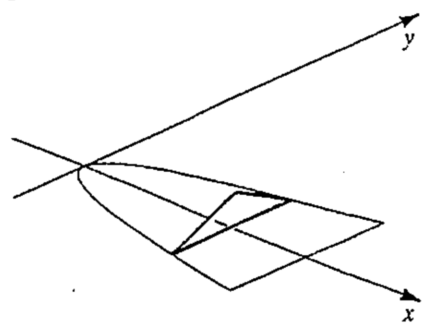
4 A quartercircle



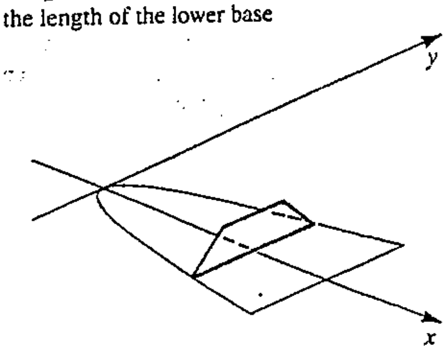
5 An equilateral triangle



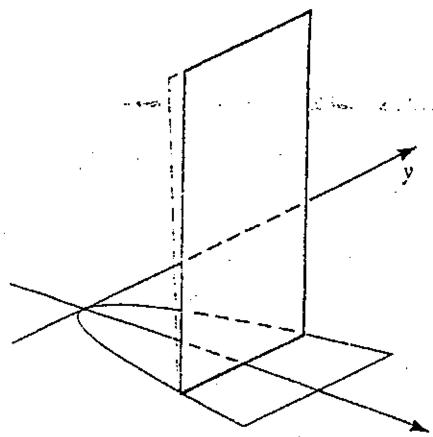
6 A triangle with height equal to $\frac{1}{4}$ the length of the base



7 A trapezoid with lower base in the xy-plane, upper ba equal to $\frac{1}{2}$ the length of the lower base, and height equ to $\frac{1}{4}$ the length of the lower base



8 A parallelogram with base in the xy-plane and height equal to twice the length of the base



9 A solid has as its base the circular region in the xy-plane bounded by the graph of $x^2 + y^2 = a^2$ with $a > 0$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a square.

10 Work Exercise 9 if every cross section is an isosceles triangle with base on the xy-plane and altitude equal to the length of the base.

11 A solid has as its base the region in the xy-plane bounded by the graphs of $y = 4$ and $y = x^2$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an isosceles right triangle with hypotenuse on the xy-plane.

12 Work Exercise 11 if every cross section is a square.

13 Find the volume of a pyramid of the type illustrated in Figure 5.39 if the altitude is h and the base is a rectangle of dimensions a and $2a$.

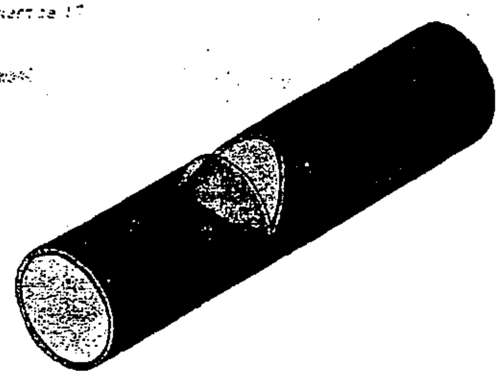
14 A solid has as its base the region in the xy-plane bounded by the graphs of $y = x$ and $y^2 = x$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is a semicircle with diameter in the xy-plane.

15 A solid has as its base the region in the xy-plane bounded by the graphs of $y^2 = 4x$ and $x = 4$. If every cross section by a plane perpendicular to the y-axis is a semicircle, find the volume of the solid.

16 A solid has as its base the region in the xy-plane bounded by the graphs of $x^2 = 16y$ and $y = 2$. Every cross section by a plane perpendicular to the y-axis is a rectangle whose height is twice that of the side in the xy-plane. Find the volume of the solid.

17 A log having the shape of a right circular cylinder of radius a is lying on its side. A wedge is removed from the log by making a vertical cut and another cut at an angle of 45°, both cuts intersecting at the center of the log (see figure). Find the volume of the wedge.

Exercise 17



18 The axes of two right circular cylinders of radius a intersect at right angles. Find the volume of the solid bounded by the cylinders.

19 The base of a solid is the circular region in the xy-plane bounded by the graph of $x^2 + y^2 = a^2$ with $a > 0$. Find the volume of the solid if every cross section by a plane perpendicular to the x-axis is an isosceles triangle of constant altitude h. (*Hint:* Interpret $\int_{-a}^{a} \sqrt{a^2 - x^2}\,dx$ as an area.)

20 Cross sections of a horn-shaped solid by planes perpendicular to its axis are circles. If a cross section that is s inches from the smaller end of the solid has diameter $6 + \frac{1}{36}s^2$ inches and if the length of the solid is 2 ft, find its volume.

21 A tetrahedron has three mutually perpendicular faces and three mutually perpendicular edges of lengths 2, 3, and 4 cm, respectively. Find its volume.

22 *Cavalieri's theorem* states that if two solids have equal altitudes and if all cross sections by planes parallel to their bases and at the same distances from their bases have equal areas, then the solids have the same volume (see figure). Prove Cavalieri's theorem.

Exercise 22